Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, First Semester
Mid-Sem Examination - 2013-2014
Analysis III

Time: 3 Hours September 10, 2013

Instructions : Check that the paper carries at least 35 marks. Maximum marks you can get is 35.

1. Let $Q : R^2 \longrightarrow R$ be any C^1 function. Let a bounded open subset of R^2 have the form
$$\begin{aligned} G &= \{(x,y) : a < x < b,\ f_1(x) < y < f_2(x)\}, \\ &= \{(x,y) : c < y < d,\ g_1(y) < x < g_2(y)\} \end{aligned}$$
where $f_1, f_2 : [a,b] \longrightarrow R,\quad g_1, g_2 : [c,d] \longrightarrow R$ are suitable continuous functions satisfying $f_1(x) < f_2(x)$ for x in (a,b) and $g_1(y) < g_2(y)$ for y in (c,d). Prove Greens theorem namely
$$\int\int_G \frac{\partial Q}{\partial x}\, dx \;\; dy = \int_{\partial G} Q\ dy$$
where ∂G is traced in the anticlock wise direction. [6]

2. Let $f : [0,\infty] \times R \longrightarrow R$ be given by
$$f(x,y) \;\; = \;\; \begin{cases} 0 \text{ if } x = 0 \text{ or } y = 0 \\ y \text{ if } 0 \le y \le \sqrt{x} \\ -y + 2\sqrt{x} \text{ if } \sqrt{x} \le y \le 2\sqrt{x} \\ 0 \qquad\quad \text{otherwise} \end{cases}$$
Extend f to $R \times R$ by demanding $f(-x,-y) = -f(x,y)$ for all x, y.

Define $G(x) = \int\limits_{-1}^{1} \ f(x,y)dy$

Show that $G'(0) \neq \int\limits_{-1}^{1} \ \frac{\partial f}{\partial x}\ (0,y)\ dy$ [8]

3. Let $f : (0,1) \times (0,1) \longrightarrow R$ be given by
$$f(x,y) \;\; = \;\; \frac{x^2 - y^2}{(x^2+y^2)^2}$$
Show that both the iterated integrals
$$\int_0^1 \left[\int_0^1 f(x,y)dx \right] dy, \;\; \int_0^1 \left[\int_0^1 f(x,y)dy \right] dx$$
exist but are not equal. [6]

4. Let $g : [a,b] \times [c,d] \longrightarrow R$ be any continuous function. You can assume that $\varphi(x) = \int_c^d g(x,y)dy$, $\psi(y) = \int_a^b g(x,y)dx$ $\varphi : [a,b] \longrightarrow R$, $\psi : [c,d] \longrightarrow R$ are continuous. Also assume that $\int\int_{[a,b]\times[c,d]} g(x,y)dxdy$ exist. Show that all the three integrals $\int\int_{[a,b]\times[c,d]} g(x,y)dxdy$, $\int_a^b \left[\int_c^d g(x,y)dy\right] dx$, $\int_c^d \left[\int_a^b g(x,y)dx\right] dy$ are equal.

[8]

5. Let $k : (a,b) \longrightarrow R$ be any bounded continuous function. Let $a < a_n < b_n < b$ and $a_n \to a$, $b_n \to b$.

(i). Show that $\lim_{n\to\infty} \int_{a_n}^{b_n} k(t)dt$ exists.

(ii). Show that the limit is independent of the sequences of a_n, b_n.

[2+2]

6. Let $g_n(x) = x(1-x^2)^n$.

(a) Find $\int_0^1 g_n(x)dx$ (b) For $f_n(x) = n\ g_n(x)$

Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ and $\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right]$ are different. [1+1]

7. (a) Let (X,d) be a metric space. $g_1, g_2, \cdots : (X,d) \longrightarrow R$ are bounded continuous functions. D be a dense subset of (X,d). Assume that $\sup_{x\in D} |\ g_k(x) - h(x)| \longrightarrow 0$ as $k \to \infty$ for some bounded function $h : D \to R$. Show that h has a continuous extension to the whole of X. [4]

(b) (Hint). For $g : (X,d) \longrightarrow R$ any bounded continuous function,
$\sup_{x\in X} |\ g(x)| \leq \sup_{y\in D} |\ g(y)|$ [1]

(c) Let $h : (0,1] \longrightarrow R$ be given by $h(x) = \sin(\frac{1}{x})$. Show that no one can find a sequence of polynomials $p_1, p_2, \cdots$ such that $0 = \lim_{n\to\infty} \sup_{x \text{ in } (0,1]} |\ p_n(x) - h(x)|$ [2]